Mirror Maps, Modular Relations and Hypergeometric Series I ◇

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Abstract. Motivated by the recent work of Kachru-Vafa in string theory, we study in Part A of this paper, certain identities involving modular forms, hypergeometric series, and more generally series solutions to Fuchsian equations. The identity which arises in string theory is the simpliest of its kind. There are nontrivial generalizations of the identity which appear new. We give many such examples – all of which arise in mirror symmetry for algebraic K3 surfaces. In Part B, we study the integrality property of certain q-series, known as mirror maps, which arise in mirror symmetry.

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1. Introduction

Physicists have recently come up with apparently different realizations of a same physical theory, using a remarkable phenomenon known as *duality* [1][2][3][4][5][6]. In a given theory, this allows one to study and compute the same physical quantity by drastically different means. Upon proper mathematical interpretation, such computations often lead to remarkable relations between mathematical objects. In Part A of this paper, we study certain functional relations, and their generalizations, arising in the recent work of Kachru-Vafa on the so-called heterotic-type II duality. The objects in question are certain automorphic forms, and the relations involved are power series identities which we loosely call "modular relations".

In Part B, we consider the mirror map for the configuration of degree p Calabi-Yau hypersurfaces in $\mathbf{P}^{p-1}$ – one for each odd prime p. We will prove that the coefficients of the mirror map $z(q)$, as a q-series, are integral. This problem goes back to the first celebrated computation of the mirror map for the quintic in $\mathbf{P}^4$ [7]. We have established for the first time the integrality of mirror maps for infinitely many cases. While we restrict our detailed discussion to the cases of the degree p hypersurface in $\mathbf{P}^{p-1}$, we will later indicate how the same technique can be applied to other known cases as well.

The technique consists of some elementary applications of:

1. The exponential version of Dwork's lemma on p-adic power series;
2. Some theorems of Dwork on hypergeometric functions;
3. An estimate using the p-adic Gamma function.

The relevance of Dwork's theory on the integrality question was suggested by Ogus.[1]

Notation: Throughout this paper, a holomorphic function f defined on a disc $|q| < r$ will be written as $f(q)$ when regarded as a power series. We denote $q\frac{df}{dq}$ by f'. The parameter q will sometimes be identified with $e^{2\pi it}$ with $Im\ t > 0$. We will say that f is a solution to a differential operator L if $Lf = 0$ in a domain where this makes sense.

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[1] We thank M. Kontsevich for communicating this idea to us.

2. Part A. Modular Relations: the first example

The identity in question is:

$$\left(\sum_{n=0}^{\infty}\frac{(6n)!}{(3n)!(n!)^3}\frac{1}{j(q)^n}\right)^2 = E_4(q). \tag{2.1}$$

Here j is the Dedekind-Klein j-function, E_4 is the Eisenstein series of weight 4 with respect to $SL(2,\mathbf{Z})$. It has a Fourier expansion

$$E_4(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \tag{2.2}$$

where $\sigma_3(n)$ is the sum of 3th powers of the divisors of n.

We discuss briefly how such an identity arises in [8]. One begins with a comparison of a certain compactification of the heterotic superstring along $K^3\times T^2$ with a type IIA superstring compactified along a Calabi-Yau variety X. The variety is a degree 12 hypersurface in weighted projective space $\mathbf{P}^4[1,1,2,2,6]$. (see [8].) This hypersurface is singled out as a candidate on the basis of matching of the *physical spectrum* of the heterotic theory with the *Hodge numbers* of the Calabi-Yau variety on the type II side. The two theories are conjectured to be physically equivalent. As a nontrivial check, some quantum couplings of the two theories were compared in a suitable boundary component in the moduli space of X. On the heterotic side, an unnormalized coupling is given by (see [9] [10])

$$F_{\tau\tau\tau} = \frac{j'(\tau)^3}{j(\tau)(j(\tau)-j(i))^2} \tag{2.3}$$

where τ is the period ratio of the T^2 on the heterotic side. The corresponding normalized coupling $\tilde{F}_{\tau\tau\tau}$ differs by an overall gauge – a holomorphic function on the upper half plane. On physical grounds, it is argued that $F_{\tau\tau\tau}/E_4$ should be equal to the quantum coupling $K_{t_1t_1t_1}$ on X in a limit when one of the Kähler moduli t_2 goes to ∞, and that t_1 should be identified with τ.

Mirror symmetry for X predicts that the normalized coupling is given by [2]

$$K_{\tau\tau\tau} = \frac{1}{w_0(x)^2}\left(\frac{dx}{d\tau}\right)^3\frac{1}{x^3(1-1728x)^2} \tag{2.4}$$

[2] Actually this is only one of two terms. The second term is not relevant in this discussion.

(see [11] [12]) where $x = x(\tau)$ is the mirror map restricted along $t_2 = \infty$, and w_0 is the holomorphic period for the mirror manifold of X. It has been observed numerically in [12]that $x(\tau) = \frac{1}{j(\tau)}$. Thus the fact that $F_{\tau\tau\tau}/E_4$ should agree with $K_{\tau\tau\tau}$ suggests the identity $w_0(x(\tau))^2 = E_4(\tau)$. This has been independently observed by the authors of [13] and by us. Computing $w_0(x)$, one gets the LHS of (2.1).

Note that the right hand side (2.1)is a modular form of weight 4 while the left hand side seems to have weight 0! But the catch is that the left hand side has only a finite radius of convergence. Thus (2.1)holds only in the common domain of definitions of both sides, not the whole upper half plane.

Eqn. (2.1) is an immediate consequence of two classical identities. Namely,

$$\begin{aligned} {}_2F_1(1/12, 5/12; 1; 1728/j)^4 =& E_4 \\ {}_2F_1(1/12, 5/12; 1; z)^2 =& {}_3F_2((1/6, 5/6, 1/2; 1, 1; z), \end{aligned} \tag{2.5}$$

the right hand side of the second equation being equal to the left hand side of (2.1)with $z = 1728/j$. The first identity in (2.5)was already known to Fricke in his work on elliptic functions [14].

3. Generalizations

It is easy to generalize (2.1)to relations involving modular forms for triangle groups. Recall that the hypergeometric equation:

$$z(1-z)y'' + (1 - \frac{3z}{2})y' - \frac{1}{16}(1 - 4\nu^2)y = 0 \tag{3.1}$$

has a unique holomorphic solution y_0 near $z = 0$ with $y_0(0) = 1$, and a solution y_1 with $y_1 = y_0 Log\ z + O(z)$. The inverse $z(q)$ of the power series $q = exp\left(\frac{y_1}{y_0}\right) = z + O(z^2)$ defines an invertible holomorphic function in a disc. The inverse is then denoted by $z(q)$ and we write $x(q) = \frac{1}{\lambda} z(\lambda q)$ for a given λ.

Proposition 3.1. *For any complex numbers λ, ν with $\lambda \neq 0$, we have*

$${}_3F_2(\frac{1}{2}, \frac{1}{2} + \nu, \frac{1}{2} - \nu; 1, 1; \lambda x(q))^2 = \frac{x'^2}{x^2(1 - \lambda x)}. \tag{3.2}$$

Note that λ is just a trivial rescaling on both sides of (3.2). Equation (2.1)is the special case with $(\lambda,\nu)=(2^6 3^3, 3^{-1})$, where the right hand side is the modular form E_4 for the group $\Gamma_0(1)$. For $(\lambda,\nu)=(2^8,2^{-2}),(2^2 3^3,2^{-1}3^{-1}),(2^6,0)$, one gets respectively

$$\begin{aligned}\left(\sum_{n=0}^{\infty}\frac{(4n)!}{n!^4}x_2(q)^n\right)^2 &= \frac{{x_2'}^2}{x_2^2(1-256x_2)}\\ \left(\sum_{n=0}^{\infty}\frac{(2n)!(3n)!}{n!^5}x_3(q)^n\right)^2 &= \frac{{x_3'}^2}{x_3^2(1-108x_3)}\\ \left(\sum_{n=0}^{\infty}\frac{(2n)!^3}{n!^6}x_4(q)^n\right)^2 &= \frac{{x_4'}^2}{x_4^2(1-64x_4)}\end{aligned} \tag{3.3}$$

where the right hand sides are weight 4 modular forms of the genus zero groups $\Gamma_0(2)+,\Gamma_0(3)+,\Gamma_0(4)+$ respectively (see [15] on notations), and x_2,x_3,x_4 are their respective hauptmoduls. It can be shown that (3.3)follows from relations of hypergeometric series similar to (2.5). We shall generalize the above identities to cases involving a class of differential equations of Fuchsian type. In some special cases, it gives relations involving modular forms for other genus zero groups which are nontriangle, hence generalizing the above identities. We now give an elementary proof of the proposition above.
Proof: From classical theory of Schwarzian equation, it follows that $x(q)$ is a solution to

$$2Qx'^2+\{x,t\}=0 \tag{3.4}$$

where

$$Q=\frac{1+(-\frac{5}{4}+\nu^2)\lambda x+(1-\nu)(1+\nu)\lambda^2x^2}{4x^2(1-\lambda x)^2}. \tag{3.5}$$

On the one hand, ${}_3F_2(\frac{1}{2},\frac{1}{2}+\nu,\frac{1}{2}-\nu;1,1;\lambda x)$ is the unique power series solution with leading term $1+O(x)$ to the differential operator ($\Theta_x=x\frac{d}{dx}$):

$$L=\Theta_x^3-\lambda x(\Theta_x+1/2)(\Theta_x+1/2+\nu)(\Theta_x+1/2-\nu). \tag{3.6}$$

On the other hand

$$L\frac{x'}{x(1-\lambda x)^{1/2}}=(1-\lambda x)^{1/2}x^2x'^{-2}\frac{d}{dt}(2Qx'^2+\{x,t\}). \tag{3.7}$$

The right hand side vanishes by virtue of the Schwarzian equation. Now (3.2)follows from uniqueness. •

3.1. remarks

It is easy to show that each of the above hauptmoduls is algebraic over $\mathbf{Q}(j)$. The explicit polynomial relation between each hauptmodul and j has been constructed in [16]. Such a polynomial relation, together with each relation in (3.3), gives yet another relation between a generalized hypergeometric function F, the j function and its derivative.

On the physics side, the above examples also arise in certain degeneration of Calabi-Yau compactification of type II strings – in much the same way (2.1)arises in degenerating a family of degree 12 hypersurfaces in $\mathbf{P}^4[1,1,2,2,6]$ [8]. The three cases with $(\lambda,\nu) = (2^8, 2^{-2}), (2^2 3^3, 2^{-1}3^{-1}), (2^6, 0)$ above correspond respectively to the following three types of Calabi-Yau varieties in weighted projective spaces: $X_8(1,1,2,2,2)$, $X_{6,4}(1,1,2,2,2,2)$, $X_{4,4,4}(1,1,2,2,2,2,2)$ (the list of integers being the weights, the subscripts being the degrees). These examples have recently been studied in [17]in the context of string duality. We expect the above modular relations will be relevant for understanding heterotic-type II duality; see [6].

4. Further Generalizations

In the interesting cases, the modular relations derived above involved three ingredients: (1) a modular function x of a suitable type, (2) a modular form E of weight 4, and (3) a power series solution w_0 of a third order differential operator L. The x satisfies a Schwarzian equation (3.4)determined by a rational function Q. The E is an algebraic expression of x, x'. And the monodromy of solutions near $x=0$ to operator L has maximal unipotency. (A consequence of this is the uniqueness of power series solution w_0.)

Given a genus zero group G and a hauptmodul $x(q)$, it's easy to show that the Schwarzian derivative

$$\{x,t\} = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2 \tag{4.1}$$

is a modular form of weight 4. Thus it takes the form $-2Qx'^2$ (the -2 is just for convenience) for some rational function Q. To obtained a relation analogous to (3.2), our proof suggests that we should construct a weight 4 modular form $E = x'^2/r(x)$ and an operator L whose monodromy has maximal unipotency at $x=0$, such that $LE^{1/2}=0$. It turns out that in Mirror Symmetry, there is an abundance of operators with maximal unipotent monodromy. The four explicit examples we have seen come in fact from the following 1-parameter families of K3 surfaces in weighted projective spaces: $X_6(1,1,1,3)$, $X_4(1,1,1,1)$,

$X_{2,3}(1,1,1,1,1)$, $X_{2,2,2}(1,1,1,1,1,1)$. The corresponding third order differential operators L in these cases are precisely the Picard-Fuchs operators. The same operators also arise from certain degenerations of Calabi-Yau threefolds; see [17]and [6].

In the following we construct a list of examples using mirror symmetry. Each of the genus zero groups and its hauptmodul are of the types considered in [15]. The corresponding rational function Q will be given. We write it in the form $Q = \frac{p}{4r^2}$ where p, r are polynomials which are relatively prime. The weight 4 modular form takes the form $\frac{x'^2}{xr(x)}$. The operator L will be the Picard-Fuchs operator obtained by degenerating a given family of K3 surfaces which are complete intersections of l hypersurfaces in a product of k projective spaces (see [18] and references therein.) The degrees of the l hypersurfaces are given by the matrix $\mathbf{d} = (d_j^{(i)})_{i=1,..,k;j=1,..,l}$. The dimensions of the ith projective space is denoted by D_i. Note that the adjunction formula requires that for each i,

$$D_i + 1 = \sum_{j=1}^{l} d_j^{(i)}. \tag{4.2}$$

The deformation parameters of this family are denoted by $x_1, .., x_k$ as defined in [18]. The following table gives a list of examples up to $k \leq 4$.) The degeneration is along the "diagonal" $x := x_1 = \cdots = x_k$. The L is an operator of Fuchsian type and is of the form

$$L = \Theta_x^3 - \sum_{i=1}^{m} \lambda_i x^i p_i(\Theta_x) \tag{4.3}$$

where the p_i are monic polynomials of degree 3. It is easy to see that L has a unique power series solution w_0 with $w_0(0) = 1$. In fact its coefficients are determined by the recursion relation:

$$A_n = \frac{1}{n^3} \sum_{i=1}^{m} p_i(n-i) A_{n-i}. \tag{4.4}$$

There is also a standard construction for $w_0(x)$ in mirror symmetry. It can be obtained by restricting the holomorphic period of the k-moduli K3 surfaces along the diagonal $x := x_1 = \cdots = x_k$. The general formula is given by

$$w_0(x) = \sum_{n_1+\cdots+n_k=n} \frac{\Pi_{j=1}^{l}(n_1 d_j^{(1)} + \cdots + n_k d_j^{(k)})!}{\Pi_{i=1}^{k} n_i!^{D_i+1}} x^n \tag{4.5}$$

where the sum is over all nonnegative $n_1, .., n_k, n$.

The modular relation we obtain takes the form (cf. (3.2))

$$w_0(x)^2 = \frac{x'^2}{xr(x)} \tag{4.6}$$

It can be proved in a completely analogous ways as for (3.2). First we check that

$$L\frac{x'}{(xr(x))^{1/2}} \equiv 0 \;\; mod \;\; 2Qx'^2 + \{x,t\}. \tag{4.7}$$

Now use the fact that $x(t)$ a hauptmodul of the genus zero group G †, and hence satisfies a Schwarzian equation given by $2Qx'^2 + \{x,t\} = 0$.

We have also included three examples (end of table below) for which we know a modular relation and the genus zero group, but we do not know the corresponding family of K3 surfaces, or if it exists.

† See [15]on notations.

degree matrices $\mathbf{d}$	diff. operators L	potential $Q(x)$	genus 0 groups G
$\begin{pmatrix} 4 \end{pmatrix}$	$\Theta_x^3 - 8x(1+2\Theta_x)(1+4\Theta_x)(3+4\Theta_x)$	$\frac{1-304x+61440x^2}{4(1-256x)^2x^2}$	$\Gamma_0(2)+$
$\begin{pmatrix} 2 & 3 \end{pmatrix}$	$\Theta_x^3 - 6x(1+2\Theta_x)(1+3\Theta_x)(2+3\Theta_x)$	$\frac{1-132x+11340x^2}{4(1-108x)^2x^2}$	$\Gamma_0(3)+$
$\begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$	$\Theta_x^3 - 8x(1+2\Theta_x)^3$	$\frac{1-80x+4096x^2}{4(1-64x)^2x^2}$	$\Gamma_0(4)+$
$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	$\Theta_x^3 - 8x(1+2\Theta_x)^3$	$\frac{1-80x+4096x^2}{4(1-64x)^2x^2}$	$\Gamma_0(4)+$
$\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$	$\Theta_x^3+36x^2(1+\Theta_x)(1+2\Theta_x)(3+2\Theta_x)$ $-2x(1+2\Theta_x)(3+10\Theta_x+10\Theta_x^2)$	$\frac{1-52x+1500x^2-6048x^3+15552x^4}{4(1-36x)^2(1-4x)^2x^2}$	$\Gamma_0(6)+$
$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$	$\Theta_x^3+x^2(1+\Theta_x)^3$ $-x(1+2\Theta_x)(5+17\Theta_x+17\Theta_x^2)$	$\frac{1-44x+1206x^2-44x^3+x^4}{4x^2(1-34x+x^2)^2}$	$\Gamma_0(6)+6$
$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$	$\Theta_x^3-32x^2(1+\Theta_x)(1+2\Theta_x)(3+2\Theta_x)$ $-2x(1+2\Theta_x)(2+7\Theta_x+7\Theta_x^2)$	$\frac{1-36x+972x^2+3712x^3+12288x^4}{4(-1-4x)^2x^2(-1+32x)^2}$	$\Gamma_0(6)+$
$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	$\Theta_x^3-3x^2(1+\Theta_x)(2+3\Theta_x)(4+3\Theta_x)$ $-x(1+2\Theta_x)(4+13\Theta_x+13\Theta_x^2)$	$\frac{1-34x+745x^2+840x^3+648x^4}{4(-1-x)^2x^2(-1+27x)^2}$	$\Gamma_0(7)+$
$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \end{pmatrix}$	$\Theta_x^3+64x^2(1+\Theta_x)^3$ $-2x(1+2\Theta_x)(2+5\Theta_x+5\Theta_x^2)$	$\frac{1-28x+396x^2-1792x^3+4096x^4}{4(1-16x)^2(1-4x)^2x^2}$	$\Gamma_0(12)+$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\Theta_x^3-4x^2(1+\Theta_x)(3+4\Theta_x)(5+4\Theta_x)$ $-2x(1+2\Theta_x)(1+3\Theta_x+3\Theta_x^2)$	$\frac{1-16x+224x^2+976x^3+3840x^4}{4(-1-4x)^2x^2(-1+16x)^2}$	$\Gamma_0(10)+$
$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$	$\Theta_x^3+36x^2(1+\Theta_x)(1+2\Theta_x)(3+2\Theta_x)$ $-2x(1+2\Theta_x)(3+10\Theta_x+10\Theta_x^2)$	$\frac{1-52x+1500x^2-6048x^3+15552x^4}{4(1-36x)^2(1-4x)^2x^2}$	$\Gamma_0(6)+$
$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\Theta_x^3+64x^2(1+\Theta_x)^3$ $-2x(1+2\Theta_x)(2+5\Theta_x+5\Theta_x^2)$	$\frac{1-28x+396x^2-1792x^3+4096x^4}{4(1-16x)^2(1-4x)^2x^2}$	$\Gamma_0(12)+$
$\begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\Theta_x^3+64x^2(1+\Theta_x)^3$ $-2x(1+2\Theta_x)(2+5\Theta_x+5\Theta_x^2)$	$\frac{1-28x+396x^2-1792x^3+4096x^4}{4(1-16x)^2(1-4x)^2x^2}$	$\Gamma_0(12)+$
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$	$\Theta_x^3-98x^3(1+\Theta_x)(2+\Theta_x)(3+2\Theta_x)$ $-x(1+2\Theta_x)(5+11\Theta_x+11\Theta_x^2)$ $+x^2(1+\Theta_x)(141+242\Theta_x+121\Theta_x^2)$	$\frac{1-32x+482x^2-3332x^3+12553x^4-27636x^5+28812x^6}{4x^2(-1+4x)^2(-1+18x-49x^2)^2}$	$\Gamma_0(14)+$
$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$	$\Theta_x^3-192x^3(1+\Theta_x)(2+\Theta_x)(3+2\Theta_x)$ $-6x(1+2\Theta_x)(1+2\Theta_x+2\Theta_x^2)$ $+4x^2(1+\Theta_x)(51+88\Theta_x+44\Theta_x^2)$	$\frac{1-36x+572x^2-5088x^3+26688x^4-78336x^5+110592x^6}{4(1-8x)^2x^2(-1+4x)^2(-1+12x)^2}$	$\Gamma_0(12\|2)+$

degree matrices $\mathbf{d}$	diff. operators L	potential $Q(x)$	genus 0 groups G
$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$	$\Theta_x^3 - 30x^3(1+\Theta_x)(2+\Theta_x)(3+2\Theta_x)$ $-x(1+2\Theta_x)(3+7\Theta_x+7\Theta_x^2)$ $+x^2(1+\Theta_x)(33+58\Theta_x+29\Theta_x^2)$	$\frac{1-20x+206x^2-336x^3+57x^4-1980x^5+2700x^6}{4x^2(-1+12x)^2(-1+2x-5x^2)^2}$	$\Gamma_0(15)+$
$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$	$\Theta_x^3 - 8x^3(3+2\Theta_x)^3$ $-4x(1+2\Theta_x)(1+2\Theta_x+2\Theta_x^2)$ $+16x^2(1+\Theta_x)(5+8\Theta_x+4\Theta_x^2)$	$\frac{1-24x+272x^2-1488x^3+4352x^4-6144x^5+4096x^6}{4x^2(-1+4x)^2(-1+12x-16x^2)^2}$	$\Gamma_0(20)+$
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\Theta_x^3 + 64x^2(1+\Theta_x)^3$ $-2x(1+2\Theta_x)(2+5\Theta_x+5\Theta_x^2)$	$\frac{1-28x+396x^2-1792x^3+4096x^4}{4(1-16x)^2(1-4x)^2x^2}$	$\Gamma_0(12)+$
$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & 0 \end{pmatrix}$	$\Theta_x^3 - 192x^3(1+\Theta_x)(2+\Theta_x)(3+2\Theta_x)$ $-6x(1+2\Theta_x)(1+2\Theta_x+2\Theta_x^2)$ $+4x^2(1+\Theta_x)(51+88\Theta_x+44\Theta_x^2)$	$\frac{1-36x+572x^2-5088x^3+26688x^4-78336x^5+110592x^6}{4(1-8x)^2x^2(-1+4x)^2(-1+12x)^2}$	$\Gamma_0(12\|2)+$
$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	$\Theta_x^3 - 8x^3(3+2\Theta_x)^3$ $-4x(1+2\Theta_x)(1+2\Theta_x+2\Theta_x^2)$ $+16x^2(1+\Theta_x)(5+8\Theta_x+4\Theta_x^2)$	$\frac{1-24x+272x^2-1488x^3+4352x^4-6144x^5+4096x^6}{4x^2(-1+4x)^2(-1+12x-16x^2)^2}$	$\Gamma_0(20)+$
$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$	$\Theta_x^3 - 14x^3(1+\Theta_x)(2+\Theta_x)(3+2\Theta_x)$ $-24x^4(2+\Theta_x)(3+2\Theta_x)(5+2\Theta_x)$ $-x(1+2\Theta_x)(4+7\Theta_x+7\Theta_x^2)$ $+x^2(1+\Theta_x)(72+106\Theta_x+53\Theta_x^2)$	$(1-22x+225x^2-1292x^3+4436x^4$ $-8304x^5+5124x^6+2016x^7+6912x^8)/$ $(4(1-8x)^2(1-4x)^2(-1-x)^2x^2(-1+3x)^2)$	$\Gamma_0(30)+$
$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\Theta_x^3 + 64x^4(2+\Theta_x)^3$ $-x(1+2\Theta_x)(4+7\Theta_x+7\Theta_x^2)$ $-8x^3(3+2\Theta_x)(18+21\Theta_x+7\Theta_x^2)$ $+x^2(1+\Theta_x)(88+122\Theta_x+61\Theta_x^2)$	$(1-22x+225x^2-1224x^3+4168x^4$ $-9792x^5+14400x^6-11264x^7+4096x^8)/$ $(4(1-8x)^2(1-x)^2x^2(1-5x+8x^2)^2)$	$\Gamma_0(28)+$
(??)	$\Theta_x^3 - 54x(1+2\Theta_x)(4+9\Theta_x+9\Theta_x^2)$ $+81x^2(1+\Theta_x)(2+3\Theta_x)(4+3\Theta_x)$	$\frac{1-24x+648x^2}{4x^2(-1+27x)^2}$	$\Gamma_0(2)-$
(??)	$\Theta_x^3 - 8x(1+2\Theta_x)(3+8\Theta_x+8\Theta_x^2)$ $+1024x^2(1+\Theta_x)(1+2\Theta_x)(3+2\Theta_x)$	$\frac{1-48x+3072x^2}{4x^2(-1+64x)^2}$	$\Gamma_0(3)-$
(??)	$\Theta_x^3 - 8x(1+2\Theta_x)(1+2\Theta_x+2\Theta_x^2)$ $+256x^2(1+\Theta_x)^3$	$\frac{1-16x+256x^2}{4x^2(-1+16x)^2}$	$\Gamma_0(3)-$

Two observations: 1. in each case above, the differential operator L always takes the

form

$$L=\Theta_x^3-\sum_{i=1}^m \lambda_i x^i(\Theta_x+i/2)(\Theta_x+i/2+\nu_i)(\Theta_x+i/2-\nu_i) \tag{4.8}$$

where the λ_i, ν_i are algebraic numbers. We denote the power series solution to L with leading coefficient 1 as $w(\lambda;\nu;x)$. 2. The corresponding hauptmodul $x(q)$ is a solution to a Schwarzian equation (3.4), hence $x(q)$ is the function defined by the inverse of the relation $q = exp\,(y_1(x)/y_0(x))$ where y_1, y_0 are suitable solutions to the ODE $y''+Qy=0$. (When the modular group is triangle, $x(q)$ is nothing but the inverse of a triangle function.)

Given $\lambda=(\lambda_1,..,\lambda_m)$, $\nu=(\nu_1,..,\nu_m)$ as in (4.8), consider the differential operator

$$\tilde{L}=\Theta_x^2-\sum_{i=1}^m \lambda_i x^i(\Theta_x+i/4+\nu_i/2)(\Theta_x+i/4-\nu_i/2). \tag{4.9}$$

The singularities of (4.9)are $\infty, 0$, which are both regular, and the roots of $1-\lambda_1 x-\cdots-\lambda_m x^m=0$, each of which is regular if its multiplicity does not exceed 2. In this case, (4.9)is Fuchsian.

The operator $\tilde{L}$ has a power series solution

$$\tilde{w}(\lambda;\nu;x)=\sum_{n=0}^\infty A_n x^n \tag{4.10}$$

determined by $A_0=1$ and the recursion relation

$$A_n=\frac{1}{n^2}\sum_{i=1}^m \lambda_i(n-i+i/4+\nu_i/2)(n-i+i/4-\nu_i/2)A_{n-i}. \tag{4.11}$$

A second solution takes the form $w(\lambda;\nu;x)Log\ x+g(x)$ where $g(x)$ is a power series determined by $g(0)=0$ and the relation

$$\tilde{L}g=-2\Theta_x w+2\sum_{i=1}^m \lambda_i x^i(\Theta_x+i/4)w. \tag{4.12}$$

Thus we define

$$x(\lambda;\nu;q)=x(q)=inverse\ of\ the\ relation\ q=x\ exp\,(g/\tilde{w})\,. \tag{4.13}$$

Theorem 4.2. *Given λ,ν as above, we have the identity*

$$w(\lambda;\nu;x)^2=\frac{x'^2}{x^2(1-\lambda_1 x-\cdots-\lambda_m x^m)} \tag{4.14}$$

This subsumes all the examples given above.

Proof: We will prove the following identities

$$
\begin{aligned}
&1.\ w(\lambda;\nu;x) = \tilde{w}(\lambda;\nu;x)^2\\
&2.\ \tilde{w}(\lambda;\nu;x)^4 = \frac{x'^2}{x^2(1-\lambda_1 x - \cdots - \lambda_m x^m)}
\end{aligned}
\tag{4.15}
$$

from which our theorem follows immediately.

To prove 1., by the uniqueness of power series solution to L, it is enough to prove that $L\tilde{w}^2 = 0$. Using $\tilde{L}\tilde{w} = 0$, we do

$$
\begin{aligned}
\Theta_x^3 \tilde{w}^2 =& 2\tilde{w}\Theta_x^3\tilde{w} + 6\Theta_x\tilde{w}\Theta_x^2\tilde{w}\\
=& \sum_{i=1}^m (2\tilde{w}\Theta_x + 6\Theta_x\tilde{w})\lambda_i x^i(\Theta_x + i/4 + \nu_i/2)(\Theta_x + i/4 - \nu_i/2)\tilde{w}\\
=& \sum_{i=1}^m \lambda_i x^i(2\tilde{w}(\Theta_x + i) + 6\Theta_x\tilde{w})(\Theta_x + i/4 + \nu_i/2)(\Theta_x + i/4 - \nu_i/2)\tilde{w}.
\end{aligned}
\tag{4.16}
$$

Combining this with the identity

$$
\begin{aligned}
&(2f(\Theta_x + i) + 6\Theta_x f)(\Theta_x + i/4 + \nu_1/2)(\Theta_x + i/4 - \nu_i/2)f\\
&\qquad\qquad = (\Theta_x + i/2)(\Theta_x + i/2 + \nu_i)(\Theta_x + i/2 - \nu_i)f^2
\end{aligned}
\tag{4.17}
$$

it follows immediately that $L\tilde{w}^2 = 0$.

To prove 2., by the uniqueness of power series solution to $\tilde{L}$, it is enough to prove that $\tilde{L}\frac{x'^{1/2}}{x^{1/2}(1-\lambda_1 x - \cdots - \lambda_m x^m)^{1/4}} = 0$. Write $A = x^{-1/2}(1-\lambda_1 x - \cdots - \lambda_m x^m)^{-1/4}$. Rewriting $\tilde{L}f$, we get

$$
\tilde{L}f = x^2(1 - \sum \lambda_i x^i)\frac{d^2 f}{dx^2} + x(1 - \sum \lambda_i x^i(1+i/2))\frac{df}{dx} - \sum \lambda_i x^i(i^2/16 - \nu_i^2/4)f. \tag{4.18}
$$

Recall that under the change of variable $f = exp\left(-\int \frac{q}{2p}\right)g$, the second order expression $p\frac{d^2 f}{dx^2} + q\frac{df}{dx} + rf$ goes into its reduced form $\frac{d^2 g}{dx^2} + Qg$ for some Q. In our case (4.18), the change of variable is simply $f = x^{-1/2}(1-\lambda_1 x - \cdots - \lambda_m x^m)^{-1/4} g = Ag$. Since $x(q)$ is the inverse of the ratio of two solutions Ag_0, Ag_1 to $\tilde{L}$, its easy to see that $x' = g_0^2/Wronskian(g_0, g_1)$. Since the wronskian is constant it follows that $x'^{1/2}$ is proportional to g_0. Thus $\tilde{L}\frac{x'^{1/2}}{x^{1/2}(1-\lambda_1 x - \cdots - \lambda_m x^m)^{1/4}}$ is proportional to $\tilde{L}Ag_0$, which is zero. This proves our claim. •

A simple computation shows that the reduced form $\frac{d^2}{dx^2}+Q(x)$ of (4.9)has

$$Q = \frac{1}{4p_2^2}(-p_1^2 + 4p_0p_2 - 2p_2p_1' + 2p_1p_2') \tag{4.19}$$

where the p_i is the coefficient of $\frac{d^i f}{dx^i}$ in (4.18).

We end this section with two other observations. The first gives the reduced form of our operator L. The second is an analogue of formula 2 above, but for Gauss' *hypergeometric series.*

Proposition 4.3. *The operator L (4.8)has the reduced form $\frac{d^3}{dx^3}+4Q(x)\frac{d}{dx}+2Q'(x)$ where Q is given by (4.19).*

Proof: A direct computation of the change of variable seems too tedious. We will thus give a more conceptual proof. Given a general third order operator $L=\sum_{i=0}^3 p_i(x)\frac{d^i}{dx^i}$, its reduced form becomes $\frac{d^3}{dx^3}+4Q(x)\frac{d}{dx}+2Q'(x)$ for some Q iff the following differential equation holds:

$$\begin{aligned} &-4p_2^3+18p_1p_2p_3-54p_0p_3^2+27p_3^2p_1'-18p_2p_3p_2'+18p_2^2p_3' \\ &-27p_1p_3p_3'+18p_3p_2'p_3'-18p_2{p_3'}^2-9p_3^2p_2''+9p_2p_3p_3''=0. \end{aligned} \tag{4.20}$$

It is easy to check that the form of this condition is independent of coordinate. That is, if we wrote L in a different coordinate z: $L=\sum_{i=0}^3 q_i(z)\frac{d^i}{dz^i}$, then its reduced form becomes $\frac{d^3}{dz^3}+4R(z)\frac{d}{dx}+2R'(z)$ for some R iff (4.20)holds with the p replaced by the q. Moreover, if L has such a reduced form in one coordinate z, it does so in any other coordinate. Thus to check condition (4.20), we choose the simplest coordinate.

In our case (4.8), we choose $z = Log\ x$ (cf. $\Theta_x=\frac{d}{dLog\ x}$). Then (4.8)becomes $L=\sum_{i=0}^3 q_i(z)\frac{d^i}{dz^i}$ with

$$\begin{aligned} q_3 =&1-\sum\lambda_i e^{iz} \\ q_2 =&-\frac{3}{2}\sum\lambda_i i e^{iz} \\ q_1 =&-\sum\lambda_i e^{iz}(3i^2/4-\nu_i^2) \\ q_0 =&-\sum\lambda_i e^{iz}(i^3/8-i\nu_i^2/2). \end{aligned} \tag{4.21}$$

Observe that we get $q_2=\frac{3}{2}q_3'$, $q_1=\frac{3}{4}q_3''+r$, $q_0=\frac{1}{8}q_3'''+\frac{1}{2}r'$ where $r(z)=\sum\lambda_i\nu_i^2e^{iz}$. Now substitue these relations together with $p_i=q_i$ into (4.20), we see that this condition holds identically. This shows that the reduced form of L becomes $\frac{d^3}{dx^3}+4Q(x)\frac{d}{dx}+2Q'(x)$ for some Q.

Now if we write $L=\sum_{i=0}^{3} q_i(x)\frac{d^i}{dx^i}$, then a simple computation gives

$$Q=\frac{1}{12q_3^2}(-q_2^2+3q_1q_3-3q_3q_2'+3q_2q_3'). \tag{4.22}$$

Also from the explicit form of L we get

$$\begin{aligned} q_3 =& xr \\ q_2 =& 3xr'/2 \\ q_1 =& xr''/2+4xs \\ q_0 =& -\sum \lambda_i x^i(i-2\nu_i)(i+2\nu_i)i/8 \end{aligned} \tag{4.23}$$

where $r(x):=x^2(1-\sum\lambda_i x^i)$, $s(x):=-\sum\lambda_i x^i(i^2/16-\nu_i^2/4)$. Now from (4.19), (4.18), we get

$$\begin{aligned} p_2 =& r \\ p_1 =& r'/2 \\ p_0 =& s. \end{aligned} \tag{4.24}$$

Substituting (4.24)into (4.19), (4.23)into (4.22), we see that the two expressions coincide. •

Consider the differential operator for the hypergeometric equation:

$$\Theta_x^2-\lambda x(\Theta_x+a)(\Theta_x+b). \tag{4.25}$$

Once again it has a solution $\tilde{w}_0$, regular at $x=0$, with leading term 1, and a solution $\tilde{w}_1$ with leading term $Log\ x$. We can define $x(q)$ as before to be the inverse of the power series relation $q=e^{\tilde{w}_1(x)/\tilde{w}_0(x)}$.

Proposition 4.4. *Let $x(q)$ be as just defined. Then*

$$_2F_1(a,b;1;\lambda x(q))^4=\frac{x'^2}{x^2(1-\lambda x)^{2(a+b)}}. \tag{4.26}$$

Proof: In the proof of the theorem above, formula 2 covers the case (when $m=1$) in which $a+b=1/2$. The argument for general a,b is completely analogous. Indeed, we check that $\frac{x'^{1/2}}{x^{1/2}(1-\lambda x)^{(a+b)/2}}$ is a solution to (4.25) and is regular at $x=0$ with leading term 1. Thus it must coincide with $_2F_1(a,b;1;\lambda x)$. •

4.1. remarks

Existence of relations involving power series solution to second and third order Fuchsian equations and modular forms, clearly suggests similar relations involving series solutions to ODEs of higher order. Remarkably, there indeed exists such generalizations. Namely it is an identity of the form

$$w(x)^2 = x'^{N-1} R(x) \tag{4.27}$$

where $w = 1 + O(x)$ is a series solution to certain Nth order Fuchsian ODE, and R is a suitable algebraic function whose singularities lie along those of the ODE. Under suitable hypothesis, the right hand side is a modular form of weight $2N-2$ of an appropriate type. This will be discussed in details in a future paper [19].

5. Part B. Integrality of Mirror Maps

The set up of the problem is as follows. Recall that the a smooth degree N Calabi-Yau hypersurface X of dimension $N-2$ has as its mirror a canonical family of toric hypersurfaces X^* with $h^{N-3,1}(X^*) = 1$ [20][7](see also [21]). This family fibers over $\mathbf{P}^1 - \{0, N^N, \infty\}$. The Picard-Fuchs equation for this family is given by

$$\left(\Theta_z^{N-1} - Nz(N\Theta_z + 1)\cdots(N\Theta_z + N - 1)\right) f(z) = 0. \tag{5.1}$$

By the Frobenius method, a basis of solutions is given by

$$\omega_i(z) = \left(\frac{1}{2\pi i}\frac{\partial}{\partial\rho}\right)^i \sum_{k\geq 0} \frac{\Gamma(N(k+\rho)+1)}{\Gamma(k+\rho+1)^N}|_{\rho=0} \tag{5.2}$$

$i = 0, .., N-1$.

Let J be the Fubini-Studi Kähler class pullbacked to X, $\mathcal{C}$ be the real Kähler cone, and $\mathcal{K} = i\mathcal{C} + H^2(X, \mathbf{R})/H^2(X, \mathbf{Z})$ the complexified Kähler cone of X. The mirror map is defined locally as a map from a domain $\mathcal{K}(r) = \{tJ | Im\ t > r\} \subset \mathcal{K}$ with $r >> 0$, to $\mathbf{P}^1$ regarded as a deformation space of complex structures for the mirror family X^*. If we let $q = e^{2\pi i t}$, then the map is given by the q-series which the inverse of the relation

$$q = exp(\frac{\omega_1(z)}{\omega_0(z)}) \tag{5.3}$$

Our problem is to give a complete proof that all the coefficients in the q-series $z(q)$ are integral. For simplicity, we will restrict to the case when N is prime.

Theorem 5.5. *For each prime N, the coefficients of the q-series $z(q)$ defined above are rational integers.*

Proof: By induction, it's easy to see that $z(q)$ is integral iff the inverse $q(z)$ is integral. Now $q(z) = exp(\frac{\omega_1(z)}{\omega_0(z)}) = ze^{\frac{g(z)}{\omega_0(z)}}$ for some $g(z) \in z\mathbf{Q}[[z]]$. We will prove that $e^{\frac{g(z)}{\omega_0(z)}} \in 1 + X\mathbf{Z}_p[[X]]$ for all prime p.

To write down $g(z)$ explicitly, let

$$\begin{aligned} D_x(m) &= \sum_{j=0}^{m-1} \frac{1}{x+j} \\ D(m) &= \sum_{i=1}^{N-1} D_{\frac{i}{N}}(m) - (N-1)D_1(m). \end{aligned} \tag{5.4}$$

Then it's easy to check that

$$g(z) = \sum_{m>0} \frac{(Nm)!}{(m!)^N} D(m) z^m. \tag{5.5}$$

6. Step 1: Dwork's Lemma

Lemma 6.6. *Let $F(X) = \sum a_i X^i \in 1 + X\mathbf{Q}_p[[X]]$. Then $F(X) \in 1 + X\mathbf{Z}_p[[X]]$ iff $F(X^p)/F(X)^p \in 1 + pX\mathbf{Z}_p[[X]]$.*

For proof, see [22]

Corollary 6.7. *Let $f(X) \in X\mathbf{Q}_p[[X]]$. Then $e^{f(X)} \in 1 + X\mathbf{Z}_p[[X]]$ iff $f(X^p) - pf(X) \in pX\mathbf{Z}_p[[X]]$.*

Proof: Write $F(X) = e^{f(X)}$ which is in $1 + X\mathbf{Q}_p[[X]]$. If $e^{f(X)} \in 1 + X\mathbf{Z}_p[[X]]$, then by Dwork's Lemma,

$$F(X^p)/F(X)^p = e^{f(X^p) - pf(X)} \in 1 + pX\mathbf{Z}_p[[X]]. \tag{6.1}$$

Write the RHS as $1 - pXG(X)$ with $G(X) \in \mathbf{Z}_p[[X]]$. Then

$$f(X^p) - pf(X) = log_p(1 - pXG(X)) = \sum_{i>0} \frac{(pXG(X))^i}{i}. \tag{6.2}$$

But $p^i/i \in p\mathbf{Z}_p$ for all $i > 0$. Thus $f(X^p) - pf(X) \in pX\mathbf{Z}_p[[X]]$.

Conversely suppose $f(X^p) - pf(X) = pXH(X)$ with $H(X) \in \mathbf{Z}_p[[X]]$. Using the fact that for $n > 0$, $ord_p \frac{p^n}{n!} = n - \frac{n-S(n)}{p-1} > 0$ where $S(n)$ is the sum of the p-adic digits of n, we see that

$$\frac{e^{f(X^p)}}{e^{pf(X)}} = e^{pXH(X)} = 1 + \sum_{n>0} \frac{p^n}{n!} X^n H(X)^n \in 1 + X\mathbf{Z}_p[[X]]. \tag{6.3}$$

Thus by Dwork's Lemma, we conclude that $e^{f(X)} \in 1 + X\mathbf{Z}_p[[X]]$. •

If we let

$$f(z) = \frac{g(z)}{\omega_0(z)} = \frac{\sum_{m>0} \frac{(Nm)!}{(m!)^N} D(m) z^m}{\sum_{m\geq 0} \frac{(Nm)!}{(m!)^N} z^m}, \tag{6.4}$$

our problem then reduces to proving that $f(z^p) - pf(z) \in pz\mathbf{Z}_p[[z]]$ for all prime p. We will first prove that $e^{f(z)} \in 1 + X\mathbf{Z}_p[[X]]$ for $p \neq N$. Then show that $f(z^p) - pf(z) \in pz\mathbf{Z}_p[[z]]$ for $p = N$.

7. Step 2: Dwork's Theorems on p-adic hypergeometric series

We'll not state the most general setting for Dwork's Theorems which require a rather long discussion. Instead we state the theorems only in the generality we need here.

Notations: Let Ω be the completion of $\bar{\mathbf{Q}}_p$, $\mathcal{O}$ the ring of integers of Ω, $\Omega^\times$ the group of units of Ω, C the set of all rational numbers which are p-integral but which are neither zero nor a negative rational integer. The following are extracted from [23]

Theorem 7.8. *[23]Let $A_0, A_1 : \mathbf{Z}_{\geq 0} \to \Omega^\times$, $g_0, g_1 : \mathbf{Z}_{\geq 0} \to \mathcal{O} - \{0\}$ be mappings satisfying the conditions that*

(i) $|A_i(0)| = 1$;

(ii) $A_i(m) \in g_i(m)\mathcal{O}$;

(iii) for all $a, \mu, s \in \mathbf{Z}_{\geq 0}$ such that $a < p$, $\mu < p^s$ we have

$$\frac{A_0(a + \mu p + mp^{s+1})}{A_0(a + \mu p)} - \frac{A_1(\mu + mp^s)}{A_1(\mu)} \in p^{s+1} \frac{g_1(m)}{g_0(a + \mu p)} \mathcal{O}. \tag{7.1}$$

Then for all $m, s, M \in \mathbf{Z}_{\geq 0}$, we have

$$H_a(m, s, M) \in p^{s+1} g_1(m) \mathcal{O} \tag{7.2}$$

where

$$H_a(m,s,M) = \sum_{j=mp^s}^{(m+1)p^s-1} \left(A_0(a+(M-j)p)A_1(j) - A_1(M-j)A_0(a+jp)\right). \quad (7.3)$$

Define a mapping $C \to C$, $x \mapsto x'$, where x' is the element such that $px' - x$ is the minimal representative in $\mathbf{Z}_{\geq 0}$ of the class of $-x \ mod\ p$. Let $\theta_1, .., \theta_N$ be elements of C. Define

$$\begin{aligned} F(t) &= \sum_{m\geq 0} \alpha(m)t^m \\ \alpha(m) &= \frac{\Pi_{i=1}^N \theta_i(\theta_i+1)\cdots(\theta_i+m-1)}{m!^{N-1}} \\ \tilde{F}(t) &= \sum_{m\geq 1} \alpha(m)D(m)t^m \\ D(m) &= \sum_{i=1}^{N-1} D_{\theta_i}(m) - (N-1)D_1(m) \\ G(t) &= \sum_{m\geq 0} \alpha'(m)t^m \\ \tilde{G}(t) &= \sum_{m\geq 1} \alpha'(m)D'(m)t^m \end{aligned} \quad (7.4)$$

where α', D' are given by the same formulas as α, D but with the θ_i replaced by θ_i'. Note that $F(t)$ is a special case of a so-called generalized hypergeometric series ${}_{N-1}F_{N-2}[\theta;\sigma;t]$, ie. it is a power series solution to the generalized hypergeometric equation

$$\left(t\frac{d}{dt}\Pi_{j=1}^{N-2}(t\frac{d}{dt}+\sigma_j-1) - t\Pi_{i=1}^{N-1}(t\frac{d}{dt}+\theta_i)\right)g(t) = 0. \quad (7.5)$$

Theorem 7.9. *[23]In the above notations, we have*

$$\frac{\tilde{G}}{G}(t^p) \equiv p\frac{\tilde{F}}{F}(t) \quad mod\ p\mathbf{Z}_p[[t]]. \quad (7.6)$$

Now assume that the prime $p \neq N$, and let $\theta_i = \frac{i}{N}$. By $(N,p)=1$, it's easy to check that the mapping $x \mapsto x'$ acts by permutation on the set $\{\theta_1, .., \theta_{N-1}\}$. This means that $G = F$ and $\tilde{G} = \tilde{F}$. We conclude from Theorem 4.1 that for $p \neq N$,

$$\frac{\tilde{F}}{F}(t^p) \equiv p\frac{\tilde{F}}{F}(t) \quad mod\ p\mathbf{Z}_p[[t]]. \quad (7.7)$$

By Corollary in section 6, we have $exp(\frac{\tilde{F}(t)}{F(t)}) \in 1 + t\mathbf{Z}_p[[t]]$. Thus $exp(\frac{\tilde{F}(kt)}{F(kt)}) \in 1 + t\mathbf{Z}_p[[t]]$ for any $k \in \mathbf{Z}_p$.

Now note that $\alpha(m) = \frac{(Nm)!}{m!^N N^{Nm}}$, which implies that $\omega_0(z) = F(zN^N)$, $g(z) = \tilde{F}(zN^N)$. So for $p \neq N$, we can conclude that $exp(\frac{g(z)}{\omega(z)}) \in 1 + z\mathbf{Z}_p[[z]]$. Thus it remains to prove $f(z^p) - pf(z) \in pz\mathbf{Z}_p[[z]]$ for $p = N$.

8. Step 3: $p = N$.

Since $\frac{(Nm)!}{m!^N}$ is a product of binomial coefficients, we have $\omega_0(z) \in \mathbf{Z}[[z]]$. So it's enough to show that $\omega_0(z)g(z^p) - p\omega_0(z^p)g(z) \in pz\mathbf{Z}_p[[z]]$. But from eqn. (5.4), it's clear that $D_x(m) \in p\mathbf{Z}_p[[z]]$ for $x = \theta_1, .., \theta_{N-1}$. Thus it's enough to show that (see eqn. (5.5)):

$$h(z) := \sum A(n)A(m)D_1(m)z^{mp+n} - p\sum A(n)A(m)D_1(n)z^{mp+n} \in pz\mathbf{Z}_p[[z]] \tag{8.1}$$

where $A(n) := \frac{(pm)!}{m!^p}$. The coefficient of z^{a+Mp} in $h(z)$ ($0 \leq a < p$, $M \in \mathbf{Z}_{\geq 0}$) is

$$L(a+Mp) := \sum_{j=0}^{M} A(a+jp)A(M-j)\left(D_1(M-j) - pD_1(a+jp)\right). \tag{8.2}$$

From eqn. (5.4), trivially we have $pD_1(a+jp) \equiv D_1(j) \ mod \ p\mathbf{Z}_p$. Thus

$$\begin{aligned} L(a+Mp) &= \sum_{j=0}^{M} A(a+jp)A(M-j)\left(D_1(M-j) - D_1(j)\right) \\ &= \sum_{j=0}^{M} D_1(j)\left(A(a+jp)A(M-j) - A(a+(M-j)p)A(j)\right) \\ &= -\sum_{s=0}^{r}\sum_{m=0}^{p^{1+r-s}-1} Y_{m,s} \quad \text{where} \\ Y_{m,s} &= \left(D_1(mp^s) - D_1([\frac{m}{p}]p^{s+1})\right) H_a(m,s,M) \\ H_a(m,s,M) &= \sum_{j=mp^s}^{(m+1)p^s-1} \left(A(a+(M-j)p)A(j) - A(M-j)A(a+jp)\right). \end{aligned} \tag{8.3}$$

for some r with $p^r > M$. The last expression for $L(a+Mp)$ is obtained by applying Lemma 4.2 in [23]. Now by writing $m = b + Rp$, $0 \leq b < p$, $R = [\frac{m}{p}]$ the integer part of $\frac{m}{p}$, we have

$$D_1(mp^s) - D_1([\frac{m}{p}]p^{s+1}) = (1 + \frac{1}{2} + \cdots + \frac{1}{Rp^{s+1} + bp^s}) - (1 + \frac{1}{2} + \cdots + \frac{1}{Rp^{s+1}}) \equiv 0 \ mod \ \frac{1}{p^s}\mathbf{Z}_p. \tag{8.4}$$

Hence

$$Y_{m,s} \in \frac{1}{p^s} H_a(m, s, M)\mathbf{Z}_p. \tag{8.5}$$

To complete the proof that $L(a + Mp) \in p\mathbf{Z}_p$, we'll prove that

$$H_a(m, s, M) \in p^{s+1}\mathbf{Z}_p \tag{8.6}$$

for $a, m, s, M \in \mathbf{Z}_{\geq 0}$, $0 \leq a < p$. For this we'll apply Theorem 1.1 with $A_0(n) = A_1(n) = A(n)$, $g_0(n) = g_1(n) = 1$. The hypotheses (i), (ii) there obviously hold. So we must check hypothesis (iii). Thus we claim that for all $a, \mu, s \in \mathbf{Z}_{\geq 0}$ such that $a < p$, $\mu < p^s$ we have

$$ord \left(\frac{A(a + \mu p + mp^{s+1})}{A(a + \mu p)} - \frac{A(\mu + mp^s)}{A(\mu)} \right) \geq s + 1. \tag{8.7}$$

To prove this we will use the p-adic Gamma function to estimate the LHS.

First we state some simple facts. Recall that

$$ord\ A(n) = ord\ (pn)! - p\ ord\ n! = S(n) \tag{8.8}$$

where $S(n)$ is the sum of the p-adic digits of n. Recall that the p-adic Gamma function is given by

$$\begin{aligned} \Gamma_p(n) =& (-1)^n \gamma(n) \quad \text{ where} \\ \gamma(n) =& \Pi_{0<j<n,\ (n,p)=1} j. \end{aligned} \tag{8.9}$$

We can write $\gamma(1 + np) = \frac{(np)!}{p \cdot 2p \cdots np}$, hence

$$(np)! = \gamma(1 + np)n!p^n. \tag{8.10}$$

Also for positive integers n, k, r, we have (see [24])

$$\Gamma_p(n + kp^r) \equiv \Gamma_p(n) \ \ mod\ p^r. \tag{8.11}$$

We apply these formulas repeatedly in the following computation:

$$
\begin{aligned}
&\frac{A(a+\mu p+mp^{s+1})A(\mu)}{A(a+\mu p)A(\mu+mp^s)}\\
=&\frac{(ap+\mu p^2+mp^{s+2})!}{(a+\mu p+mp^{s+1})!^p}\frac{(\mu+mp^s)!^p}{(\mu p+mp^{s+1})!}\frac{(a+\mu p)!^p}{(ap+\mu p^2)!}\frac{(\mu p)!}{\mu!^p}\\
=&\gamma(1+ap+\mu p^2+mp^{s+2})(a+\mu p+mp^{s+1})!^{1-p}p^{a+\mu p+mp^{s+1}}\\
&\times\gamma(1+\mu p+mp^{s+1})^{-1}(\mu+mp^s)!^{p-1}p^{(\mu+mp^s)(-1)}\\
&\times\gamma(1+ap+\mu p^2)^{-1}(a+\mu p)!^{p-1}p^{(a+\mu p)(-1)}\\
&\times\gamma(1+\mu p)\mu!^{1-p}p^{\mu}\\
=&\gamma(1+ap+\mu p^2+mp^{s+2})\Pi_{i=1}^a(i+\mu p+mp^{s+1})^{1-p}\\
&\times\gamma(1+\mu p+mp^{s+1})^{1-p}(\mu+mp^s)!^{1-p}p^{(\mu+mp^s)(1-p)}p^{a+\mu p+mp^{s+1}}\\
&\times\gamma(1+\mu p+mp^{s+1})^{-1}(\mu+mp^s)!^{p-1}p^{(\mu+mp^s)(-1)}\\
&\times\gamma(1+ap+\mu p^2)^{-1}\Pi_{i=1}^a(i+\mu p)^{p-1}\gamma(1+\mu p)^{p-1}\mu!^{p-1}p^{\mu(p-1)}p^{(a+\mu p)(-1)}\\
&\times\gamma(1+\mu p)\mu!^{1-p}p^{\mu}\\
=&\frac{\gamma(1+ap+\mu p^2+mp^{s+2})\gamma(1+\mu p)^p}{\gamma(1+\mu p+mp^{s+1})^p\gamma(1+ap+\mu p^2)}\Pi_{i=1}^a(i+\mu p+mp^{s+1})^{1-p}\Pi_{i=1}^a(i+\mu p)^{p-1}\\
\equiv&\frac{\gamma(1+ap+\mu p^2+mp^{s+2})\gamma(1+\mu p)^p}{\gamma(1+\mu p+mp^{s+1})^p\gamma(1+ap+\mu p^2)}(\Pi_{i=1}^a(i+\mu p)+O(p^{s+1}))^{1-p}\Pi_{i=1}^a(i+\mu p)^{p-1}\\
\equiv&\frac{\gamma(1+ap+\mu p^2+mp^{s+2})\gamma(1+\mu p)^p}{\gamma(1+\mu p+mp^{s+1})^p\gamma(1+ap+\mu p^2)}(1+O(p^{s+1}))\\
=&\frac{\Gamma_p(1+ap+\mu p^2+mp^{s+2})\Gamma_p(1+\mu p)^p}{\Gamma_p(1+\mu p+mp^{s+1})^p\Gamma_p(1+ap+\mu p^2)}(1+O(p^{s+1}))\\
=&\frac{(\Gamma_p(1+ap+\mu p^2)+O(p^{s+2}))\Gamma_p(1+\mu p)^p}{\Gamma_p(1+ap+\mu p^2)(\Gamma_p(1+\mu p)+O(p^{s+1}))^p}(1+O(p^{s+1}))\\
\equiv&1+O(p^{s+1})
\end{aligned}
\tag{8.12}
$$

Thus we conclude that

$$
\begin{aligned}
&ord\left(\frac{A(a+\mu p+mp^{s+1})}{A(a+\mu p)}-\frac{A(\mu+mp^s)}{A(\mu)}\right)\\
=&ord\ \frac{A(\mu+mp^s)}{A(\mu)}+ord\left(\frac{A(a+\mu p+mp^{s+1})A(\mu)}{A(a+\mu p)A(\mu+mp^s)}-1\right)\\
\geq&S(\mu)+S(m)-S(\mu)+s+1\geq s+1.
\end{aligned}
\tag{8.13}
$$

This completes our proof.

9. Other cases and applications

It turns out that the above technique for studying the integrality property for the degree p hypersurfaces in $\mathbf{P}^{p-1}$ is also applicable in other cases with some minor modifications. For example, it can be verified that in numerous cases of Calabi-Yau threefolds, such a technique applies.

To see how this is done, let's consider for example the case of the degree 10 hypersurface in $\mathbf{P}^4[1,1,1,2,5]$. The Picard-Fuchs equation in this case is given by

$$\big(\Theta_z^4 - 80z(1+10\Theta_z)(3+10\Theta_z)(7+10\Theta_z)(9+10\Theta_z)\big)\, f = 0. \tag{9.1}$$

First we rescale z by $z \mapsto z/80\cdot 10^4$. The equation becomes

$$\left(\Theta_z^4 - z(\frac{1}{10}+\Theta_z)(\frac{3}{10}+\Theta_z)(\frac{7}{10}+\Theta_z)(\frac{9}{10}+\Theta_z)\right) f = 0. \tag{9.2}$$

Then the arguments in Parts I and II above apply for all $p \neq 2,5$ and with the data $\theta_1 = \frac{1}{10}, \theta_2 = \frac{3}{10}, \theta_3 = \frac{7}{10}, \theta_4 = \frac{9}{10}$.

To treat $p = 2,5$, we apply a similar argument as in Step 3 above together with an estimate on the coefficients of an appropriate generalized hypergeometric series using the $p-adic$ gamma function. This argument goes through without a hitch, and hence prove that the mirror map in this case is integral.

Finally we note that the integrality property of the mirror map is crucial for us in our previous study on some arithmetic properties of the quantum Yukawa coupling [16]. We plan to further investigate these properties.

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